An Exploration of Linear Algebraic Models of Musical Spaces

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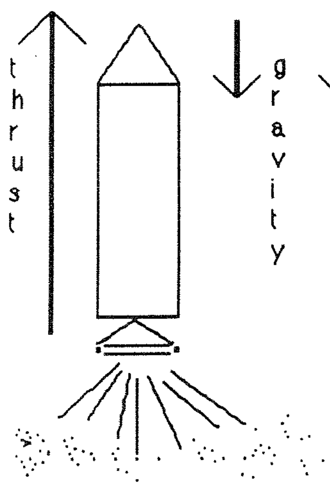
The fields of music and mathematics have long been closely associated. With the early inquiries into the properties of music by the Greeks, the importance of mathematics was discovered. Pythagoras revealed the strong link between musical intervals and mathematical proportions. Since then many theorists have invoked mathematics to justify or illuminate their musical hypotheses. Mathematics has aided theories of tuning and temperament, theories of consonance and dissonance, and has been the basis for many theories of harmony during the 18th and 19th centuries. Recently, the use of mathematics has assisted in the analysis of contemporary music and the deepening of the understanding of musical processes in earlier music. The following discussion will focus on the applicability of linear algebra as a potential for modelling musical structures.

Linear algebra is a branch of mathematics which is built on the concept that algebraic and geometric relationships are essentially interchangeable. Consequently, an algebraic formula or theorem can be formulated geometrically (i.e. graphically) in order to yield a new perspective or a more intuitive approach. Similarly, a geometric structure can be viewed algebraically to simplify operations and calculations. By using linear algebra, ideas from the rich literature of mathematical-musical models can be examined for their geometric as well as their algebraic implications. In addition, new mathematical-musical relationships or operations may be found.

Figure 1. Vectors in physics



The primary structures of linear algebra are vectors and vector spaces. Vectors can be various types of objects which, when combined in a strictly linear fashion, form a vector space. In mathematics vectors can represent such entities as force, direction, position, systems of linear equations, and numerical functions, among other things. For example, a physicist might use vectors to represent the total number of forces that are acting upon a given object, such as a rocket. Thus a vector in an upward direction would be used to represent the thrust of the rocket while downward vectors would symbolize the forces of gravity and friction or air resistance (see Figure 1). The physicist can then illustrate these vectors by drawing arrows on a Cartesian axis in which the length of each arrow represents the magnitude of the force and the direction of the arrow shows the direction of the force. The use of such a vector representation also allows the physicist to manipulate the forces mathematically in order to determine the net force or to calculate the force and direction which would be required to put the rocket into orbit.

In music, one might wish to use pitches, pitch-classes (pcs), pc sets, dynamics, attack-points, durations, or other musical elements as the basis for describing vectors. It should be noted that the concept of vector in this discussion differs slightly from the concept of "interval vector" as used by Forte, Morris and others. Forte's interval vector basically has the structure of a list. Its function is to catalogue the total number of interval classes which appear in a particular pc set.[1] For our purposes a vector will be any entity which can be described with a value of magnitude and a value of direction. For example, the musical interval C4—E4 can be represented by a vector. It has direction (upward) and magnitude (4, assuming a semitone represents the unit magnitude). By including other parameters such as duration or volume, the vector representation can be refined to produce a subtler sense of direction and magnitude.

Once the object-vectors have been chosen, it becomes possible to construct a vector space within which these objects can be manipulated. The creation of a vector space depends upon a strict combination of the elements which are chosen as vectors. The formal definition for a vector space follows:[2]

Definition

A real vector space U is a non-empty set of elements for which two algebraic operations are defined as follows:

There is an operation which assigns to any elements u and v in U an element denoted by u+v and called the sum of u and v (this operation is called addition).

There is an operation which assigns to any element u in U and any real number a an element denoted by a*u

[1] Actually, this list could be represented as a vector in the mathematical sense. If one could imagine a six-dimensional Cartesian coordinate system in which each axis represented a different interval class, an interval vector could be represented mathematically by plotting a point according to the values in the interval vector and drawing an arrow from the origin (0,0,0,0,0,0) to the plotted point. However, the interval vector was originally designed as a list or catalogue.

[2] Adapted from George D. Mostow and Joseph H. Sampson, *Linear Algebra* (New York: McGraw-Hill Book Company, 1969), 15.

or au and called their scalar product (this operation is called scalar multiplication). These operations are required to obey the following rules:

For any elements u, v, w in U,

u+v = v+u (commutative law);
(u+v)+w = u+(v+w) (associative law).

U contains an element denoted by **0** and called zero such that

u+**0** = u for every element u in U.

For any real numbers a and b,

a(u+v) = au+av (distributive law);
(a+b)u = au+bu (distributive law);
a(bu) = (ab)u;
0*u = **0** and **1***u = u

This definition operates regardless of the objects chosen to be represented as vectors. A concrete example using specific vector representations and their combinations will serve to illuminate some of the ramifications of vector spaces.

Let us take as our set of vectors ordered triples of real numbers of the form (x_1,x_2,x_3). We begin by defining the operations of addition and scalar multiplication on these vectors. The sum of two ordered triples (x_1,x_2,x_3) and (y_1,y_2,y_3) can be defined by the formula

$$(x_1,x_2,x_3) + (y_1,y_2,y_3) = (x_1+y_1,x_2+y_2,x_3+y_3).$$

Given a real number c and an ordered triple (x_1,x_2,x_3), the scalar product of c and (x_1,x_2,x_3) can be defined by the formula

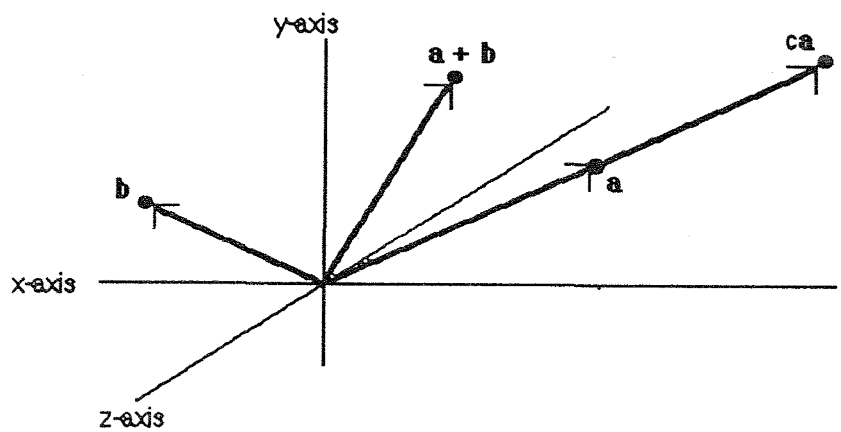
$$c * (x_1,x_2,x_3) = (cx_1,cx_2,cx_3).$$

Once addition and scalar multiplication are defined, it becomes much more convenient to refer to our vectors in simpler terms. Thus, instead of writing (x_1,x_2,x_3) and (y_1,y_2,y_3) we can use **x** and **y** (the bold print signifies that the objects are vectors instead of simple numbers). To verify that our definitions of addition and scalar multiplication do indeed form a vector space, it must be proven that they satisfy the necessary rules described above (e.g. commutativity, associativity, etc.). For example, we can show that

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1+y_1,x_2+y_2,x_3+y_3) \text{ (from our definition)} \\ &= (y_1+x_1,y_2+x_2,y_3+x_3) \text{ (since real numbers are associative)} \\ &= \mathbf{y} + \mathbf{x} \text{ (again, from our definition of addition).} \end{aligned}$$

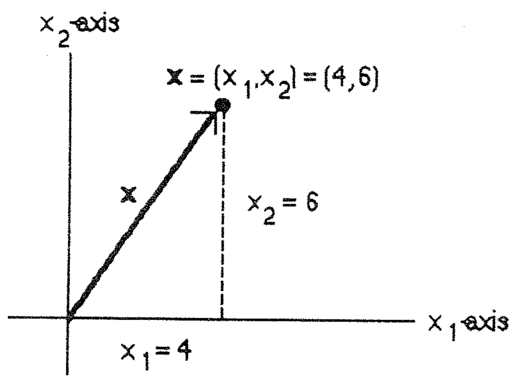
Thus, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ which satisfies the first rule of a vector space. Verification of the remaining rules is left to the reader.

Figure 2. Geometric representation of two vectors



In addition to considering this newly formed vector space from an algebraic viewpoint, we can also create a geometric representation. In a three-dimensional euclidean space (which can be described with the use of a Cartesian coordinate system) an ordered triple can be represented by a single point. Consequently, we can envision our object **x** as a vector (depicted by an arrow) which extends along each of the three axes to a certain distance from the point (0,0,0) (depending on the values of x_1, x_2, and x_3). Figure 2 illustrates a geometric representation of two vectors (vectors **a** and **b**), their vector sum (vector **a** + **b**), and a scalar product (the product of scalar c and vector **a**). In order to make this illustration more concrete, the reader may wish to verify the results by assigning specific real values to the ordered triples **a** and **b** and the scalar c (e.g. **a** = (1,7,5), **b** = (-2,-1,5) and c = 2.5). Figure 2 also suggests other concepts that might be useful, namely vector angle (the angle created between two vectors) and vector distance (the mathematical distance between two vectors).

Figure 3. Vector in two-dimensional euclidean space



To derive a formula for vector length it will be fruitful to begin with an intuitive approach. First, one can imagine a vector in two-dimensional euclidean space, say $\mathbf{x} = (x_1,x_2) = (4,6)$ (see Figure 3). We can find the length of the single vector **x**, notated $|\mathbf{x}|$ (sometimes called the absolute value of the vector), by using the Pythagorean theorem. Thus, $|\mathbf{x}|^2 = x_1^2 + x_2^2$. In three-dimensional euclidean space the Pythagorean theorem can be extended as follows: $|\mathbf{x}|^2 = x_1^2 + x_2^2 + x_3^2$. In fact, in n-dimensional space $|\mathbf{x}|^2 = x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2$. To extend our notion of length to include the concept of distance between two vectors, we shall now define what will be called the canonical inner product.[3]

[3]This is called the canonical inner product because it is not the only inner product that could be defined here. The definition of inner product must be sufficiently general to incorporate vector spaces other than those in euclidean space (such as a vector space of numerical functions). Consequently, to be

Definition

Let **u** and **v** be vectors in euclidean n-dimensional space. Then the number

$$<\mathbf{u}|\mathbf{v}> = u_1v_1 + u_2v_2 + \dots + u_nv_n$$

is called the canonical inner product of **u** and **v**; the non-negative real number

$$|\mathbf{u}| = \sqrt{<\mathbf{u}|\mathbf{u}>}$$

is called the length of the vector **u**. The distance between **u** and **v** is defined to be $|\mathbf{u} - \mathbf{v}|$.[4]

It should be noted that this definition of inner product indeed satisfies our intuition about the length of a single vector as well as producing an algebraic formulation for the distance between vectors.

It is also possible to find the angle between vectors. In order to avoid complex and lengthy derivations, it will simply be stated that a unique angle θ can be found such that[5]

$$<\mathbf{u}|\mathbf{v}> = |\mathbf{u}|*|\mathbf{v}| \cos\theta \text{ or } \theta = \cos^{-1}(<\mathbf{u}|\mathbf{v}>/|\mathbf{u}|*|\mathbf{v}|)$$

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accurate, an inner product on a real vector space is defined as a function which assigns to any ordered pair of vectors **u**, **v** a real number, denoted by $<\mathbf{u}|\mathbf{v}>$ which satisfies four conditions: 1. $<\mathbf{u}|\mathbf{u}> \geq 0$; 2. $<\mathbf{v}|\mathbf{u}> = <\mathbf{u}|\mathbf{v}>$; 3. $<a\mathbf{u}|\mathbf{v}> = a<\mathbf{u}|\mathbf{v}>$ for any scalar a; 4. $<\mathbf{u}+\mathbf{u}'|\mathbf{v}> = <\mathbf{u}|\mathbf{v}> + <\mathbf{u}'|\mathbf{v}>$. Other inner products (such as $<\mathbf{u}|\mathbf{v}> = \mathbf{u}_1\mathbf{v}_1 + 10\mathbf{u}_2\mathbf{v}_2 + 4\mathbf{u}_3\mathbf{v}_3$) can be chosen which satisfy these conditions, but only the canonical inner product makes intuitive sense in euclidean space. In other spaces, however, there may not be a specific inner product which "makes sense". Thus, it is necessary to have a general set of guidelines by which a proper inner product (or several inner products) can be chosen.

[4]Mostow and Sampson, 21.

[5]Mostow and Sampson, 23.

By creating the concepts of distance, angle, and inner product, a new space has been formed. "An inner product space H is a vector space with which a definite inner product is associated."[6] In the field of linear algebra the inner product space is a relatively low-level concept. This concept becomes the groundwork for discussions which involve orthogonal and orthonormal vectors and subspaces (those which are conceptually at right angles to one another or which are at right angles and have length 1), linear dependence or independence of a vector space, vector space bases, matrix algebra, determinants, and eigenvalues and eigenvectors. However, in this initial exploration of linear algebraic properties we will limit our discussion to the concepts associated with inner product spaces. It will now be profitable to consider some musical examples in order to illustrate some of the possibilities of a linear algebraic model.

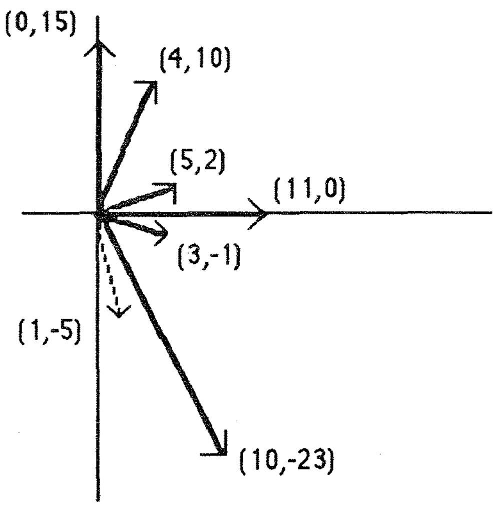
Let us take as our set of vectors real ordered pairs in two-dimensional euclidean space. Thus, our vector space U consists of elements $\mathbf{u} = (u_1,u_2)$. These elements can be constructed to describe musical pitches in time. Consequently, we will assign to u_1 values of attack points from a piece of music with the unit value representing the smallest duration in the excerpt. Negative attack point values will represent musical events that have occurred before the specific event being considered. We can assign to u_2 pitch numbers such that 0 = C4, 1 = C-sharp4, -1 = B3, etc. Thus, if we think of our unit durational value as the quarter note, the vector (3,-5) would represent the pitch G3 which is attacked 3 quarter notes after our specific point of consideration. This space is very similar to Lewin's GIS_3.[7] It differs in that it uses pitch space instead of pc space in order to make this presentation more clear. If we define an inner product to be the canonical inner product as above, we can introduce the notion of vector distance and vector angle.

Let us take as a musical example the opening notes of the third movement of the *Piano Variations*, op. 27 by Webern. The first six pitches are E-flat5, B3, B-flat4, D4, C-sharp2, and C4. A

[6] Mostow and Sampson, 42.

[7] David Lewin, *Generalized Musical Intervals and Transformations* (New Haven: Yale University Press, 1987), 37-44.

Figure 4. Angles and distances between vectors



representation of these pitches in the vector space described above would give us the following vectors: (0,15), (3,-1), (4,10), (5,2), (10,-23), (11,0). Figure 4 shows a graphic representation which visually illustrates the angles and distances between the vectors. We can now algebraically calculate the angles and distances between any of these vectors. For example, the distance between the first and second vectors ("first" and "second" refer to time location, i.e. order along the x-axis) is

$$\begin{aligned}|\mathbf{u} - \mathbf{v}| &= |(0,15) - (3,-1)| \\ &= |(-3,16)| \\ &= \sqrt{<(-3,16)|(-3,16)>} \\ &= \sqrt{-3 * -3 + 16 * 16} \\ &= \sqrt{9 + 256} \\ &= \sqrt{265} \approx 16.28.\end{aligned}$$

The angle between the first and second vectors is $\theta = \cos^{-1}(<\mathbf{u}|\mathbf{v}>/|\mathbf{u}|*|\mathbf{v}|) \approx 108^{o}$. The distance between the first vector and the penultimate vector (i.e. between the highest and lowest notes of the example) is $|\mathbf{u} - \mathbf{v}| \approx 39.29$. The angle between these two vectors is $\theta \approx 156^{o}$.

This data confirms some of our intuitions about the spatial distance or angle between pitches. Since the penultimate pitch was much lower and occurred later than the second pitch, we would expect its vector distance to be greater when measured from the first vector. Similarly, as Figure 4 suggests, the angle is also larger between the extreme vectors than between the first and second vectors. However, note that if the time of the second pitch was sooner and the pitch was slightly lower, say (1,-5), then the angle would be $\theta \approx 168^{\circ}$ which is larger than the angle between the extreme vectors. This preserves the notion that a vector which is remote in pitch and close in time location creates a greater angle than the same pitch at a distant time location.[8]

As was noted above, the pitch component of our two-dimensional example was represented in an integer pitch space. It could also have been represented in a mod12 pc space. Graphically, this would yield a Cartesian system which includes only the numbers 0, 1, 2, .., 11 on the y-axis. Care must be taken when computing the distances and angles in this representation. When normally adding or multiplying pcs, the mod12 function is applied immediately after each operation. When computing distance and angle, however, the mod12 function cannot be used. For example, say we have the pitches C4 and E4 which are struck simultaneously at the beginning of a musical excerpt. They would be represented by (0,0) and (0,4), respectively. Intuitively, we would expect the distance between the vectors to be 4. By using the formula for distance and applying mod12 arithmetic we get the following result:

[8]It should be emphasized that the distances and angles computed here were based on the initial selection of C4 as the 0 pitch. A different point of origin causes the angles between vectors to change while the distances between the vectors do not change (the reader can visualize this by mentally moving the point of origin in Figure 4). Thus, if vector angle is to be used, the point of origin should be placed at a relatively significant pitch/attack-point in the excerpt under consideration.

$$
\begin{aligned}
|\mathbf{u} - \mathbf{v}| &= |(0,0)-(0,4)| \\
&= |(0 \bmod 12, -4 \bmod 12)| \\
&= |(0,8)| \\
&= \sqrt{0 + 64} = 8.
\end{aligned}
$$

However, by excluding the mod12 arithmetic, we get

$$
\begin{aligned}
|\mathbf{u} - \mathbf{v}| &= |(0,0)-(0,4)| \\
&= |(0,-4)| \\
&= \sqrt{0 + 16} = 4
\end{aligned}
$$

which is what we expect.

The distances and angles which are calculated in vector space are different from the concept of distance (i.e., interval) as discussed by Lewin[9] and Morris.[10] Lewin and Morris are concerned with distances between individual elements of each vector. For example, Lewin discusses the interval between each pitch and between each time location in his discussion of GIS_3, but does not combine these distances into a single value. By creating a vector space, however, the total distance and angle between vectors can be calculated which takes into account every parameter that is described in the space. In other words, increases and decreases in each parameter of a vector affect the total distance and angle of one vector from another. For example, consider the dyad C4—E4 discussed above. If we change the attack point element of the E4 to indicate that it is sounded three quarter notes after the C4, it can be represented by the vector (3,4) (assuming the quarter note to be the unit durational value). We can, of course, find the distance between each of the individual parameters simply by subtracting. Thus, the distance between the pitches is still 4 and the distance between the attack points is now 3. However, we can also use the formula above

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[9]Lewin, 16-20, 25-26, 74-75.

[10]Robert D. Morris, *Composition with Pitch-Classes: A Theory of Compositional Design* (New Haven: Yale University Press, 1987), 37, 62-64, 69-70.

to compute a composite distance which merges both parameters of each vector. By substituting (0,0) and (3,4) into our distance formula, we get 5 which shows in a single value that (3,4) is farther from (0,0) than is (0,4).

From the examples presented above, it can be seen that an increase in the complexity of the representation of the musical vectors will not greatly increase the complexity of the calculations. We could, for example, include parameters of sustain-time (durational value) and dynamics in addition to attack point and pitch value in our vector representations. Dynamics might be represented in a scale from 0 (*ppp*) to 7 (*fff*) and durational values could be represented using the same scale as the attack points. Thus a single note could be represented as a four-dimensional vector such that (1,-5,7,3) would signify the note G3 which is struck 1 time unit after the point of consideration and held for 3 time units at a dynamic level of fff. The distance or "composite interval" between this pitch and a pitch (5,-10,4,2) would be:

$$
\begin{aligned}
d &= \sqrt{((1 - 5)^2 + (-5 - (-10))^2 + (7 - 4)^2 + (3 - 2)^2)} \\
&= \sqrt{(16 + 25 + 9 + 1)} \\
&= \sqrt{51} \approx 7.14.
\end{aligned}
$$

It might even be possible to devise a scale for timbres which could be included as a vector parameter. To do this, the concept of distance between specific timbres would have to be defined (perhaps through analysis of wave-forms or partials).

This initial presentation of a linear algebraic model of musical elements has served to demonstrate the fundamental properties of linear algebra and their potential to aid in the analysis of music. It has been designed to introduce a new mathematical model that has the potential for creating interesting musical interpretations.

To close, two possible directions for further extension of the above principles will be presented. One extension involves using vectors other than translations of individual pitches into the various euclidean spaces. For example, we could choose as our set of vectors all melodic fragments or pc sets of six pitches in length (hexachords). Each pitch in these sets could still be described in multiple parameters as described above. Consequently we could construct a vector of vectors. Thus, the vector ((1,-5,7,3), (2,8,6,4), (6,4,6,2), (7,14,2,1), (10,-25,2,2), (12,6,1,0)) would be a complete representation of a hexachord which is actually a vector of six vectors which each represent a pitch of varying duration, attack point, and dynamics. Distance and angle calculations between such hexachords would remain essentially the same except that the distance/angle between corresponding pitches would need to be evaluated before the total distance/angle could be calculated. This would create the possibility of measuring the distance between hexachords, melodies or different permutations of a 12-tone row.

A second extension is the organization of "vector classes." A vector class would consist of all vectors which lie a specific distance or a specific angle from a given vector. Various vectors in a piece could be analyzed and labelled according to their location in a specific vector class. Vectors belonging to a particular class relative to some other vector could be compared and contrasted. As a brief example, in our 2-dimensional inner product space described above (i.e., vectors represented by (attack point, pitch number)), the pitches (4,-3), (3,4), (0,5), and (-5,0) (among others) would all belong to the same vector distance class from (0,0) (namely distance class 5).

The applications that a linear algebraic system brings to the field of musical analysis are rich. The value of applying a mathematical system such as linear algebra can undoubtedly be observed from this initial presentation. One of the advantages of such a mathematical approach is the ability to manage large numbers of musical parameters with relatively uncomplicated mathematical expressions. This approach also brings to music the concepts of vector distance and vector angle. These operations allow the calculation of composite intervals which give the distance between musical elements of multiple parameters. Through the application of mathematical systems to music, new perspectives on the behavior of musical systems will surely be discovered.